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Matrix groups with independent spectra

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Abstract

The article deals with triangularizability of a group of matrices over an algebraically closed field F with characteristic 0 under the assumption that the spectra of elements of the group satisfy an independency condition on their multiplicative orders and transcendental independency. Let p be a prime number and let matrix A be similar to a triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s$. If for $i \neq j$ the orders of λ_i and λ_j are finite with greatest common divisor dividing p and $\mu_1, \ldots, \mu_s$ are transcendently independent over $\mathbf{Q}$, we say that the matrix A has the *p-property*. The main result in this paper is that every matrix group consisting of matrices with the 2-property is triangularizable, which is a generalization of the result for a group with spectra in the set $\{1, -1\}$ (see [1]). Some remarks on general prime p are also given. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

We first show where the definition of the p-property comes from. If every matrix A of an irreducible matrix group G is similar to a triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s$ where for $i \neq j$ the orders of λ_i and λ_j are finite numbers without a common divisor and $\mu_1, \ldots, \mu_s$ are transcendently independent over $\mathbf{Q}$, then

$$\det : G \to F$$

is a homomorphism of groups with kernel K consisting of unipotent matrices. The normal subgroup K is triangularizable by the celebrated Kolchin theorem. In the

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proof of the main theorem, one can see how the triangularizability of K affects the triangularizability of G (we use commutativity of the irreducible parts of the group K). The “invention" of the p-property then does not look so odd, because we want in case of $p = 2$ the kernel of $\det^2$ to be a subgroup consisting of matrices with eigenvalues 1 and -1. Clearly, in the case of a general p-group, G would not be triangularizable (see examples in [1]).

We now proceed with some technical lemmas.

2. Preliminaries

Let us first introduce the following notation. For an element $\lambda \in F$ with finite multiplicative order, we will denote its order by $|\lambda|$, i.e.

$$|\lambda| = \min\{t \in \mathbf{N} \mid \lambda^{\mathbf{t}} = \mathbf{1}\}.$$

Lemma 2.1. *Let $q \in \mathbf{N}$ and $\lambda, \mu \in F$. If the greatest common divisor $d(|\lambda|, |\mu|)$ divides q, then $d(|\lambda^q|, |\mu^q|) = 1$.*

Proof. Let us denote $|\lambda| = t$, $|\mu| = u$, $|\lambda^q| = r$ and $|\mu^q| = s$. First we determine the numbers r and s. It is easy to verify that

$$r = \frac{t}{d(t, q)}$$

and

$$s = \frac{u}{d(u, q)}.$$

Since $d(t, u)$ divides q, the numbers s and r are co-prime. This completes the proof. $\square$

Definition 2.1. Let p be a prime number and let the matrix A be similar to a triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s$. If for $i \neq j$ the orders of λ_i and λ_j are finite with greatest common divisor dividing p and $\mu_1, \ldots, \mu_s$ are transcendently independent over $\mathbf{Q}$, we say that the matrix A has the *p-property*. We will use this term for a single matrix or a set of matrices all having this property.

Let us denote for the rest of the paper

$$Z_p = \{\lambda \in F \mid \lambda^p = 1\}.$$

Definition 2.2. A matrix A of the form $A = DP$ where D is a diagonal matrix and P is a permutation matrix is called a *monomial matrix*. A group consisting of monomial matrices is called a *monomial group*.

For a monomial group there is an epimorphism

$$\phi : G \to P_G,$$

where P_G is the group of all permutation matrices P (see Definition 2.2) associated with the matrices from G. We will often use the notation $P_A = \phi(A)$.

Lemma 2.2.
(1) *If* $\det^p(A) = 1$ *for a matrix A with the p-property, then* $\sigma(A) \subset Z_p$.
(2) *If a monomial matrix A where* P_A *is a permutation matrix given by permutation* π *has the p-property, then the permutation* π *consists only of transpositions and p-cycles. If* $p = 2$, *then every transposition in* π *gives us a block with eigenvalues* 1 *and* -1 *in matrix A. If* $p > 2$, *then every p-cycle in* π *gives us a block with eigenvalues* $\sqrt[p]{1}$ *in the matrix A and* π *has at most one transposition.*

Proof. (1) Let the matrix A have the p-property and $\det^p(A) = 1$. Then

$$\lambda_1^p \cdots \lambda_r^p \mu_1^p \cdots \mu_s^p = 1.$$

If $0 = |\lambda_1| \cdots |\lambda_r|$, then we get

$$\mu_1^{op} \cdots \mu_s^{op} = 1.$$

As $\mu_1, \ldots, \mu_s$ are transcendently independent, it follows that $s = 0$. Since by Lemma 2.1, the orders $|\lambda_1^p|, \ldots, |\lambda_r^p|$ are co-prime, it is easy to see using the equation $\lambda_1^p \cdots \lambda_r^p = 1$ that in fact $\lambda_1^p = \cdots = \lambda_r^p = 1$ and $\sigma(A) \subset Z_p$.

(2) Suppose that π has a cycle of length n. By permuting the basis we can rearrange π to include the cycle $(123 \ldots n)$ and thus A has the form

$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix},$$

where

$$B = \begin{bmatrix} 0 & \cdots & \cdots & 0 & d_1 \\ d_2 & \ddots & & & 0 \\ 0 & d_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & d_n & 0 \end{bmatrix}.$$

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the matrix B and denote $d = d_1 \cdots d_n$. Then we get

$$\lambda_i^n = d$$

for all $i = 1, 2, \ldots, n$. If d is transcendental over $\mathbf{Q}$, then λ_i are transcendental and therefore have infinite order. But on the other side λ_i are algebraically dependent $(\lambda_i^n = \lambda_j^n)$ and A does not have the p-property.

We can therefore assume that the orders of λ_i are finite. It follows that:

$$d^{|\lambda_i|} = \lambda_i^{n|\lambda_i|} = 1,$$

so that $|d| = m$ divides $|\lambda_i|$ for all $i = 1, 2, \dots, n$ and therefore $m \in \{1, p\}$. Let ν be a primitive solution of the equation $x^{mn} = 1$. Then ν^n is a primitive solution of the equation $x^m = 1$ and we can write $d = (\nu^n)^k$. Since the order $|d|$ is m, k and m must be co-prime. Let λ be a solution of equation $x^n = d$. Then

$$\lambda^{mn} = d^m = 1,$$

so $\lambda = \nu^s$. Since

$$\nu^{sn} = \lambda^n = d = \nu^{kn}$$

m divides $s - k$ and therefore

$$s = k + lm,$$

where l ranges over a complete remainder system modulo n. Let $r = |\lambda|$. As

$$1 = \lambda^r = \nu^{(k+lm)r},$$

mn has to divide $(k + lm)r$. Since k and m are co-prime, m divides r, $r = tm$ and n divides $(k + lm)t$.

Let n contain a prime factor q. Then $q|(k + lm)t$ for all l. If $n > 2$, there are $l_1 \neq l_2$, $0 \leqslant l_1, l_2 < n$ such that q does not divide $(k + l_i m)$ since otherwise we could find such l_1, l_2 such that q divides $(k + l_i m)$ and q does not divide $(l_2 - l_1)$. From this we would get $q|(l_2 - l_1)m$ which implies $q|m$ and $q|k$. This is a contradiction since k and m are co-prime.

From the above we conclude that $n = 2$ or A has two eigenvalues with orders containing the prime factor q which implies $q = p$ and then $n = p^j$. If $j > 1$, from above we see that p^2 divides t which is again a contradiction.

Thus we have proved that $n = 2$ or $n = p$.

For the second part of (2) let us first deal with case $p = 2$. If $p = 2$, then $n = 2$. We have already seen that $d^2 = 1$. We have to exclude the possibillity $d = -1$. It is obvious since $\lambda_{1,2} = \pm\sqrt{-1}$ have order 4.

Let now $p > 2$ and $n = p$. Then every solution of the equation $\lambda^p = d$ has order dividing p^2. If $d \neq 1$, then order of λ is neither 1 nor p (as $\lambda^p = d \neq 1$) so that all solutions have orders p^2 which is a contradiction and the lemma is proved.

For the last statement in (2) one can easily verify that every transposition gives an eigenvalue with even order and therefore only one is permitted. $\square$

3. On monomial groups with the p-property

As we are concerned with monomial matrix groups let us state some remarks on previous results connected to this subject. The letter G will denote a monomial matrix group with the p-property. For a matrix A we will often make no distinction between P_A and its associated permutation π.

Remark 1. If $p \neq 2$ and G contains a matrix A with a transposition in its P_A, then P_{A^p} is a permutation matrix of a transposition and $-1 \in \sigma(A^p)$. We have the equivalence

$$P_G \text{ contains no transposition} \quad \Leftrightarrow \quad -1 \notin \sigma(G).$$

Remark 2. By the discussion given below, one could see that transpositions are possible only in cases $n = 2$ or $n = 3$ with $p = 3$. In both cases we have examples:

Case $n = 2$:

$$G_2 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| (ab)^p = 1 \right\} \cup \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \middle| (ab)^p = 1 \right\}.$$

Case $n = 3$, $p = 3$:

$$G_3 = \left\{ DP \mid P \in S_3,\ D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},\ abc = 1,\ a, b, c \in Z_3 \right\}.$$

Let $G \subset \mathrm{GL}_n(F)$ be irreducible with $n > 2$ and $p > 2$. Let P_G contain a transposition. After conjugation with a permutation matrix we can assume that the transposition $\tau = (12)$ is in P_G. Irreducibility of the group G implies transitivity of the permutation group P_G and we can find a matrix $Y \in G$ such that $P_Y(1) \notin \{1, 2\}$ and thus

$$(12) \neq P_Y \tau P_Y^{-1} \in P_G.$$

This way we find another transposition $\tau' \neq \tau$ in P_G. If τ and τ' are disjoint, we get a matrix with two transpositions in P_G which is a contradiction. The remaining possibility is that the product $\tau\tau'$ is a 3-cycle which means $p = 3$. Let us now analyze the case $p = 3$ and $n > 3$. We can assume that P_G contains the transposition $\tau = (12)$ and a permutation π with a 3-cycle (otherwise we would need another disjoint transposition for irreducibility). By conjugation with a suitable permutation from P_G we make π to be of the form $(1bc)\ldots$ If $2 \notin \{b, c\}$ by conjugation with permutation π we either get $\tau' = (b2) \in P_G$ (if π fixes 2) and $(12)(b2) = (12b) \in P_G$ or with a form τ disjoint transposition τ', which is a contradiction. Thus we can assume that the 3-cycle (123) is contained in P_G. Because P_G is transitive, we find a permutation $\rho \in P_G$ such that $\rho(1) = 4$. Since otherwise conjugation with ρ would give us a transposition disjoint with τ', we have $\rho(2) \in \{1, 2\}$. If $\rho(2) = 1$ then $\rho = (142)\ldots$, $(12)\rho = (14)\ldots$ and thus $(14) \in P_G$. We get

$$(14)\pi = (1234\ldots)\ldots,$$

which is a contradiction by Lemma 2.2. As the assumption $(14) \in P_G$ yields a contradiction in the case $\rho(2) = 2$, we get $\rho = (14j)\ldots$ where $j \neq 1, 2$. This implies

$$(12)\rho = (14j2\ldots)\ldots$$

which is again a contradiction by Lemma 2.2.

We now give a criterion for irreducibility of monomial groups by which one can get irreducibility of the above groups G_2 and G_3. Let us first state a lemma.

Lemma 3.1. *Let $G \subset \mathrm{GL}_n(F)$ have transitive group P_G and let $V \leqslant F^n$ be a G-module. Then there exists a vector $v = (v_1, \ldots, v_n) \in V$ with $v_i \neq 0$ for all i.*

Proof. Asssume that $v = (v_1, \ldots, v_{k-1}, 0, v_{k+1}, \ldots, v_n) \in V$. Since P_G is transitive there is a vector $w \in V$ with $w_k \neq 0$. For $n \in \mathbf{N}$ large enough vector $nv + w$ has at least one more nonzero component as vector v, so inductively we reach the desired vector. $\square$

For a monomial group *G*, with D_G denote the subgroup of all diagonal matrices.

Proposition 3.2. *Let $G \subset \mathrm{GL}_n(F)$ be a monomial group with transitive P_G. If the linear span of D_G is n-dimensional, then G is irreducible.*

Proof. If $D_1, \ldots, D_n \in D_G$ are linearly independent and $v \in F^n$ a vector with nonzero components (we assume this by Lemma 3.1), then it is easy to check that $D_1 v, \ldots, D_n v$ span F^n. $\square$

By the above proposition the groups G_2 and G_3 from Remark 2 are irreducible.

Definition 3.1. Let *G* be a linear group on a vector space *V*. Then the group *G* is *imprimitive* if there exists such a decomposition $V = V_1 \oplus \cdots \oplus V_r$, $r > 1$, that for every $g \in G$ and $i \leqslant r$ there is $j \leqslant r$ with the property $g(V_i) = V_j$. The subspaces V_k are *blocks of imprimitivity*.

The extreme case of imprimitivity is monomiality, where the blocks are one-dimensional.

Lemma 3.3. *Let F be an algebraically closed field, $G \subset M_n(F)$ an irreducible group and $K \lhd G$ an abelian subgroup such that the quotient G/K is abelian group. If $n > 1$, then G is imprimitive group.*

Proof. Let *Z* be the center of the group *G*. Since *G* is an irreducible group and *F* is algebraically closed $Z = G \cap FI$. If $K \subset Z$, then *G* is nilpotent and by Theorem 24 (see [2, p. 60]) monomial.

If $K \not\subset FI$, then by Lemma 7 (see [1, p. 11]) *G* is imprimitive. $\square$

Let *G* be a matrix group with the *p*-property. Then $\det^p : G \to F$ is an homomorphism of groups. Let *K* denote the kernel of this homomorphism. By Lemma 2.2

$$K = \{A \in G \mid \sigma(A) \subset Z_p\}.$$

Proposition 3.4. *Let G be an irreducible monomial matrix group with the p-property. If $p > 2$ we assume in addition that matrices in P_G are without transpositions (the latter assumption is necessary in cases described in Remark* 2), *then $G = K$ or G is one-dimensional.*

Proof. If G is a diagonal matrix group, then G is one-dimensional. Thus we assume that G contains nondiagonal matrices. Then P_G is a nontrivial group. Since all elements of the group P_G have order p, P_G is a p-group and thus has nontrivial center Z. If Z contains a matrix of the form

$$U = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix},$$

where I is an identity matrix and B is a permutation matrix of p-cycles, then it is easy to see that the subspace associated with the block I is an invariant subspace for G. Thus we can find (if we suitably rearrange the basis) a matrix $U \in Z$ of the form

$$P_0 = \begin{bmatrix} C & & & \\ & C & & \\ & & \ddots & \\ & & & C \end{bmatrix},$$

where C is the permutation matrix of the p-cycle. One can easily show that every other matrix $X \in P_G$ has the block-form

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{bmatrix}$$

with $X_{ij} = \varepsilon_{ij} C^{k_{ij}}$, $k_{ij} \geqslant 0$, where $[\varepsilon_{ij}]_{i,j}$ is a permutation matrix.

Let $A \in \phi^{-1}(P_0)$. Since A_0 consists of $(p \times p)$-blocks associated with p-cycles in P_0 by Lemma 2.2 $A_0 \in K$. If $A \in G$ is a diagonal matrix also AA_0 consists of $(p \times p)$-blocks so $AA_0 \in K$ and $A \in K$. Since A^p is a diagonal matrix for every $A \in G$, $A^p \in K$ and $\sigma(A^p) = \{1\}$. If $A \notin K$, we can find $\lambda_1 \in \sigma(A)$ with $|\lambda_1| = p^2$. Since every $p \times p$-block in A gives us only eigenvalues in Z_p, the eigenvalue λ_1 must be on the diagonal part of the matrix A. According to the block-structure of P_G we can assume the following structures of matrices A, A_0 and AA_0:

$$A = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_p & \\ & & & & \ddots \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & \cdots & 0 & d_1 & \\ d_2 & \ddots & & 0 & \\ & \ddots & & \vdots & \\ & & d_p & 0 & \\ & & & & \ddots \end{bmatrix},$$

$$AA_0 = \begin{bmatrix} 0 & \cdots & 0 & d_1\lambda_1 & \\ d_2\lambda_2 & \ddots & & 0 & \\ & \ddots & & \vdots & \\ & & d_p\lambda_p & 0 & \\ & & & & \ddots \end{bmatrix}.$$

By Lemma 2.2 we get $\lambda_1 d_1 \cdots \lambda_p d_p = 1$. As $d_1 \cdots d_p = 1$ we get $\lambda_1 \cdots \lambda_p = 1$ and $\lambda_1^p \cdots \lambda_p^p = 1$. Since orders $|\lambda_2|, \ldots, |\lambda_p|$ cannot be p^2 by the p-property of matrix A, they are all equal to p and we get $\lambda_1^p = 1$ which is a contradiction. This completes the proof. □

Remark 3. Let us now assume that G is not necessarily irreducible and not all the matrices in G are diagonal. If the center Z of P_G, which is again nontrivial, contains a matrix

$$U = \begin{bmatrix} C & & & \\ & C & & \\ & & \ddots & \\ & & & C \end{bmatrix},$$

we proceed as in the proof of Proposition 3.4 and get $G = K$. Otherwise we find a matrix $U \in Z$:

$$U = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}$$

with B consisting of p-cycles. If we decompose a matrix $M \in P_G$ according to the above block decomposition of U:

$$M = \begin{bmatrix} X & Y \\ Z & U \end{bmatrix}$$

and since M and U commute, we get the condition

$$(B - I)Z = 0.$$

If $Z \neq 0$, we can find a vector e from the basis such that

$$Be = e,$$

which is a contradiction since B has no fixed points in our basis. Similarly we get $Y = 0$. This way we see that the group G is of the form

$$G = \left\{ \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \middle| \, X_1 \in G_1,\ X_2 \in G_2 \right\},$$

where G_1 and G_2 are monomial groups of smaller dimension than G.

Now we can see inductively that a monomial group G with the p-property can be put in the form

$$G = \begin{bmatrix} D & 0 \\ 0 & K \end{bmatrix},$$

where D is a diagonal group and $\sigma(K) \subset Z_p$.

According to the conclusions of the Proposition 3.4 we restrict ourselves to the case of an irreducible matrix group $G \subset \mathrm{GL}_n(F)$ with prime exponent p. In this case the group G is nilpotent and is therefore automatically monomial (see [2, Theorem 24, p. 60]). Since G is irreducible its degree is a power of p, $n = p^k$. If $p = 2$, G is one-dimensional (see [1]). The question which arises is: What can be said about general p? In [1] one can find an example of G of degree p. By using tensor products, we can construct groups with exponent p and arbitrary degree p^k.

The following proposition shows that in the case $p = 3$ and $k = 2$ such a tensor product is the only possibility.

Proposition 3.5. *Let $G \subset \mathrm{GL}_9(F)$ be an irreducible matrix group with exponent* 3. *Then G is conjugate to a tensor product $H \otimes K$, where H, K are subgroups of the group*

$$C_3 = \left\{ DC^k \mid k = 0, 1, 2,\ D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},\ abc = 1\ a, b, c \in Z_3 \right\}$$

and

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Proof. The group G is clearly a monomial group with the 3-property. Since P_G is a homomorphic image of the group G, P_G has exponent 3. In the proof of Proposition 3.4 one can see that $P_d = C \otimes I \in Z(P_G)$ and all matrices of P_G are of the form $[\varepsilon_{ij} C^{k_{ij}}]_{i,j}$ where $[\varepsilon_{ij}]_{i,j}$ is a permutation matrix whose order divides 3. Thus we find that every matrix $P \in P_G$ takes the form

$$P = \begin{bmatrix} C^{k_1} & & \\ & C^{k_2} & \\ & & C^{k_3} \end{bmatrix} (I \otimes C^l). \tag{1}$$

Since P_G is transitive we find a matrix

$$\begin{bmatrix} & & I \\ C^m & & \\ & C^n & \end{bmatrix} \in P_G.$$

After conjugation with the matrix

$$\begin{bmatrix} I & & \\ & C^{2m} & \\ & & I \end{bmatrix}$$

we get $P_c = I \otimes C \in P_G$. As $P_d \in P_G$ we can find a matrix $X \in G$ of the form

$$\begin{bmatrix} d_1^1 &&&&&&&& \\ & d_2^1 &&&&&&& \\ && d_3^1 &&&&&& \\ &&& d_1^2 &&&&& \\ &&&& d_2^2 &&&& \\ &&&&& d_3^2 &&& \\ &&&&&& d_1^3 && \\ &&&&&&& d_2^3 & \\ &&&&&&&& d_3^3 \end{bmatrix} P_d.$$

From $X^3 = I$ we get $d_1^i d_2^i d_3^i = 1$ and thus after the conjugation with the matrix

$$\begin{bmatrix} 1 &&&&&&&& \\ & d_2^1 &&&&&&& \\ && d_2^1 d_3^1 &&&&&& \\ &&& 1 &&&&& \\ &&&& d_2^2 &&&& \\ &&&&& d_2^2 d_3^2 &&& \\ &&&&&& 1 && \\ &&&&&&& d_2^3 & \\ &&&&&&&& d_2^3 d_3^3 \end{bmatrix}$$

we can assume that $P_d \in G$. (Note that a conjugation with a diagonal matrix does not change associated group P_G.) By (1) we can write every $X \in G$ as

$$X = \begin{bmatrix} D_1 C^{k_1} && \\ & D_2 C^{k_2} & \\ && D_3 C^{k_3} \end{bmatrix} (I \otimes C^l) \tag{2}$$

with D_i diagonal matrices. If we assume that $l = 1$ and denote $A_i = D_i C^{k_i}$, we get $A_3 A_2 A_1 = I$. Since $X P_d^s \in G$ we see that

$$D_3 C^{k_1+s} D_2 C^{k_2+s} D_3 C^{k_3+s} = I \tag{3}$$

for $s = 0, 1, 2$. Let us denote the action of matrix C on the set of diagonal matrices by D^C $(= CDC^{-1})$. Then condition (3) is equivalent to

$$D_3 D_2^{C^{m+s}} D_3^{C^{n+2s}} = I, \tag{4}$$

where $m = k_3$ and $n = k_2 + k_3$. If we replace s by $s + 1$ in Eq. (4) and then multiply this equation by the inverse of Eq. (4) we get

$$\left(D_2^C D_2^{-1}\right)^{C^{m+s}} \left(D_1^{C^2} D_1^{-1}\right)^{C^{n+2s}} = I. \tag{5}$$

This implies

$$\left(D_2^C D_2^{-1}\right)^{C^{2s}} \left(D_1^{C^2} D_1^{-1}\right)^{C^{n-m}} = I. \tag{6}$$

Hence $D_2^C D_2^{-1}$ is a scalar matrix $(1/\omega)I$ and thus

$$D_2 = \beta D(\omega),$$

where

$$D(\omega) = \begin{bmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{bmatrix}.$$

From Eq. (5) we get $D_1 = \alpha D(\omega)$ and finally from (4) $D_3 = \gamma D(\omega)$. It follows that the associated diagonal matrix D_X for a matrix X is of the form

$$D_X = D(\omega) \otimes \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$$

and thus

$$X = \left(D(\omega) \otimes \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix} \right) \begin{bmatrix} C^{k_1} & & \\ & C^{k_2} & \\ & & C^{k_3} \end{bmatrix} (I \otimes C). \tag{7}$$

If $l \neq 1$ in form (2), we multiply the matrix with a matrix of form (7) and apply our conclusions. It comes out that every $X \in G$ can be written as

$$X = \left(D(\omega) \otimes \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix} \right) \begin{bmatrix} I & & \\ & C^m & \\ & & C^n \end{bmatrix} (C^k \otimes C^l). \tag{8}$$

It is now sufficient to show that $3 \mid m, n$. We suppose the opposite. Since $P_d, P_c \in P_G$ we can assume that $k = 0$ and $l = 0$ and choose

$$A = \begin{bmatrix} \alpha D(\omega) & & \\ & \beta D(\omega) & \\ & & \gamma D(\omega) \end{bmatrix} \begin{bmatrix} I & & \\ & C^m & \\ & & C^n \end{bmatrix} \in G,$$

where $m, n \in \{1, 2\}$. We can also find a matrix $F \in G$ of the form

$$F = \begin{bmatrix} \delta D(\vartheta) & & \\ & \varepsilon D(\vartheta) & \\ & & \varphi D(\vartheta) \end{bmatrix} \begin{bmatrix} & & I \\ I & & \\ & I & \end{bmatrix}.$$

From $F^3 = I$ we get

$$\delta\varepsilon\varphi = 1. \tag{9}$$

It is well known (see [3, Corollary 1, p. 100]), that a matrix group $G \subset \mathrm{GL}_n(F)$ over an algebraically closed field F is irreducible if and only if its linear span is the algebra of all $n \times n$ matrices (see [3]). The only matrices in G whose linear combinations have nonzero entries at places 11, 22, 33 are those of the form

$$X = \begin{bmatrix} \lambda D(\psi) & & \\ & \mu D(\psi) & \\ & & \nu D(\psi) \end{bmatrix} \begin{bmatrix} I & & \\ & C^k & \\ & & C^l \end{bmatrix}. \tag{10}$$

From the fact $(FA)^3 = I$, it follows that

$$\alpha\beta\gamma(\omega\vartheta)^m = 1 \tag{11}$$

and similarly from $(FA^2)^3 = I$, we get

$$\alpha^2\beta^2\gamma^2(\omega^2\vartheta)^{2m} = 1. \tag{12}$$

From Eqs. (12) and (11), it then follows that

$$\omega^m = 1,$$

which gives us

$$\omega = 1.$$

If we take a matrix of form (10), where $m = n = 0$, then the matrix AX has the property established for the matrix A and thus

$$\psi = 1.$$

Since all entries at places 11, 22, 33 are equal for all matrices in the group G, it cannot be irreducible which is a contradiction. This completes the proof. $\square$

Remark 4. In [4, p. 291] we find the 3-generator group $G = B(3, 3)$ with order 3^7. Let us denote $G' = [G, G]$. On examination of the group G one can show that $[G', G] = Z(G)$ and therefore the center $Z(G)$ is a 3-element cyclic group. Since $Z(G)$ is cyclic, G has a faithful irreducible representation ρ with character χ, and the equation $[G', G] = Z(G)$ implies that for $g \in G - Z(G)$ there exists $x \in G$ such that $[g, x] = g^{-1}x^{-1}gx = z$, where z is a nontrivial element of $Z(G)$. Let $\rho(z) = aI$. Then

$$\chi(g) = \chi(x^{-1}gx) = \chi(gz) = a\chi(g).$$

Since $a \neq 1$ it follows that $\chi(g) = 0$. Let n be the degree of the representation ρ. As χ has the norm equal to 1, we get

$$n^2 = \chi(1)^2 = |G/Z(G)| = 3^6.$$

So ρ has degree 27. One can verify that $[C_3', C_3]$ is trivial. Therefore, if the representation ρ were a tensor product of two lower-dimensional representations, then $[G, G'] = 1$. So ρ is not a tensor product.

Remark 5. It is easy to see that in case of arbitrary prime number p every irreducible group $G \subset \mathrm{GL}_p(F)$ with exponent p is conjugate to a subgroup of the group

$$C_p = \left\{ DP \mid D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_p \end{bmatrix} \lambda_i \in Z_p,\ \lambda_1 \cdots \lambda_p = 1\ P = C^l \right\},$$

where C is the permutation matrix associated with cycle $(12\ldots p)$.

4. Main theorem

Theorem 4.1. *Let $G \subset M_n(F)$ be a group of matrices over algebraically closed field F with characteristic zero with the* 2*-property* $(p = 2)$*. Then G is triangularizable.*

Proof. Clearly we can assume that G is irreducible. We have to show that G is one-dimensional.

Let us first show that G is a monomial group. We observe the homomorphism of groups

$$\det{}^2 : G \to F^*.$$

We have already seen that $K = \operatorname{Ker} \det^2$ consists of matrices with eigenvalues $1, -1$. By Clifford's theorem (see [2, p. 9]) we can decompose $F^n = L_1 \oplus \cdots \oplus L_t$, where each L_i is irreducible K-module. Since a group of matrices with eigenvalues $1, -1$ is triangularizable (see [4]), $\dim L_i = 1$ and K is diagonalizable and therefore commutative. By [3, p. 6, Lemmas 1 and 2] there exists a system of imprimitivity

$$F^n = Q_1 \oplus \cdots \oplus Q_t,$$

where the stabilizer G_i of Q_i is a primitive irreducible group. Since G_i satisfy the conditions of the theorem, we can find normal abelian groups $K_i \lhd G_i$ such that G_i/K_i are abelian. By Lemma 3.3 we see that $\dim Q_i = 1$ and G is indeed a monomial group.

By Proposition 3.4 we know that $G = K$ or G is one-dimensional. In both cases G is commutative and therefore one-dimensional. This completes the proof. $\square$

Proposition 4.2. *Let G be as in the previous theorem without* **Q***-transcendental eigenvalues and assume that G has already been triangularized. For $X \in G$ we denote with* $\operatorname{diag}_i(X)$ *the i*th *diagonal entry of the matrix X and* $D_i = \operatorname{diag}_i(G)$*. Then for $i \neq j$ an arbitrary pair $\lambda \in D_i$ and $\mu \in D_j$ satisfy the condition* $d(\lambda, \mu) \leqslant 2$ (*i.e. the condition on orders holds* "*all over*" *the group G not just matrixwise*, *which was the original assumption*).

Proof. Let us choose $X, Y \in G$ with $\operatorname{diag}_i(X) = \lambda$, $\operatorname{diag}_j(X) = \nu$, $\operatorname{diag}_i(Y) = \vartheta$, $\operatorname{diag}_j(Y) = \mu$, $|\lambda| = p$, $|\nu| = q$, $|\vartheta| = r$ and $|\mu| = s$.

We already know that for every matrix $W \in G$ the eigenvalues of W^2 have co-prime orders. For

$$Z = (X^q Y^r)^2$$

we get $\operatorname{diag}_i(Z) = (\lambda^2)^q$ and $\operatorname{diag}_j(Z) = (\mu^2)^r$. Since $|(\lambda^2)^q| = |\lambda^2|$, $|(\mu^2)^r| = |\mu^2|$, $d(\operatorname{diag}_i(Z), \operatorname{diag}_j(Z)) = 1$ the orders $|\lambda^2|$ and $|\mu^2|$ are co-prime and therefore $d(|\lambda|, |\mu|) \leqslant 2$. $\square$

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References

[1] G. Cigler, On matrix groups with finite spectra, Linear Algebra Appl. 286 (1999) 287–295.
[2] D.A. Suprunenko, Soluble and Nilpotent Linear Groups, AMS, Providence, RI, 1963.
[3] D.A. Suprunenko, Matrix Groups, AMS, Providence, RI, 1976.
[4] B. Huppert, Endliche Gruppen I, Springer, Berlin, 1967.